Development of an approximate method for quantum optical models and their pseudo-Hermicity

Ramazan Koç*
Department of Physics, Faculty of Engineering University of Gaziantep, 27310 Gaziantep, Turkey
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An approximate method is suggested to obtain analytical expressions for the eigenvalues and eigenfunctions of the some quantum optical models. The method is based on the Lie-type transformation of the Hamiltonians. In a particular case it is demonstrated that $E \times \varepsilon$ Jahn-Teller Hamiltonian can easily be solved within the framework of the suggested approximation. The method presented here is conceptually simple and can easily be extended to the other quantum optical models. We also show that for a purely imaginary coupling the $E \times \varepsilon$ Hamiltonian becomes non-Hermitian but $P\sigma_0$-symmetric. Possible generalization of this approach is outlined.

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INTRODUCTION

It is well known that the rotating wave approximation (RWA) is a useful method in determination of the eigenvalues and associated eigenfunctions of the various quantum optical Hamiltonians. The approximation gives accurate results when the frequency associated with the free evaluation of the system is essentially bigger than the transmission frequencies induced by the interaction between subsystem or external source. In quantum physics the application of the RWA usually leads to symmetry breaking: the representation space of the whole system is then divided into invariant subspaces, which strongly simplifies the mathematical complexity of the problem and usually provides the exact solution of the Hamiltonian.

The simplest model which describes a two-level atom interacting with a single mode cavity field is the Jaynes-Cummings (JC) model [1]. A considerable attention has been devoted to the interaction of a radiation field with atoms since the paper of Dicke [2]. Such system is commonly termed as the Dicke model. In spite of its simplicity, the whole spectrum of the Dicke Hamiltonian can not be obtained exactly and usually it has been treated in the framework of RWA. Besides its solution with RWA, in some papers an attempt is made to go beyond the RWA [3]. The continual integration methods are based on variational principles. The perturbative approach [4, 5] leads to more complicated mathematical treatments and the theory converges only for a certain relationship between parameters of the Hamiltonian. In a more recent study, Klimov and his co-workers [6] have developed a general perturbative approach to quantum optical models beyond the RWA, based on the Lie-type transformation.

The Jahn Teller (JT) interaction [7] is one of the most fascinating phenomena in modern physics and chemistry, providing a general approach to understanding the properties of molecules and crystals and their origins. This phenomena has inspired of the most important recent scientific discoveries, such as the concept of high temperature superconductivity. The JT interaction is an example of electron-phonon coupling. Therefore it seems that the RWA can be applied for solving the JT problems. Most of the JT Hamiltonians are more complicated than the Dicke Hamiltonian. At present, a few of them (i.e. $E\otimes\beta$, $E\otimes\epsilon$) has been analyzed in the framework of quasi-exactly solvable problem [8, 9] or isolated exact solvability [10, 11, 12, 13, 14, 15], both provide finite number of exact eigenvalues and eigenfunctions in the closed form. In this paper we devise a novel method for solving JT Hamiltonians, as well as other quantum optical Hamiltonians in the framework of RWA. It will be shown that the eigenvalues and the associated eigenfunctions can be obtained in the closed form when the coupling constant is smaller than the natural frequency of the oscillator.

The method described here includes a part of the motivation provided by the existence of the connection between JT Hamiltonians and Dicke Model. Here we concentrate our attention to the solution of the $E \otimes \epsilon$ JT Hamiltonian. Its solution has been treated previously by many authors [10, 11, 12, 16, 17, 18]. We develop a new approximation method which is based on the similarity transformation. The method introduced here is the same as the RWA which has been usually used to solve Dicke Hamiltonian. An interesting and somewhat simpler form of the JT Hamiltonian is obtained by RWA.

Other purpose of this paper is to show that for some purely imaginary couplings the $E \otimes \epsilon$ JT Hamiltonian becomes non-Hermitian but its low-lying part of the spectrum is real. It will be shown that the non-Hermitian Hamiltonian is not PT-invariant [19, 20, 21, 22, 23] , but it is pseudo-Hermitian [24, 25, 26, 27, 28].

In the following section, we shall demonstrate our procedure on the $E \otimes \epsilon$ JT Hamiltonian. We present a trans-

formation procedure and we obtain approximate form of the $E\otimes\epsilon$ JT Hamiltonian. We show that Hamiltonian can be transformed in the form of the Dicke type Hamiltonians. We also obtain explicit expressions for the eigenstates and eigenvalues of the JT Hamiltonian. In section 3, we discuss the pseudo-Hermicity of the Hamiltonian. Finally we summarize our results.

METHOD AND SUMMARY OF THE PREVIOUS RESULTS

The well-known form of the $E\otimes\varepsilon$ JT Hamiltonian describing a two-level fermionic subsystem coupled to two boson modes has obtained by Reik[12] is given by

$$H=\omega\left(a_1^+a_1+a_2^+a_2+1\right)+\omega_0\sigma_0+\kappa[(a_1+a_2^+)\sigma_++(a_1^++a_2)\sigma_-], \tag{1}$$

where ω_0 is the level separation, ω is the frequency of the oscillator and κ is the coupling strength. The Pauli matrices $\sigma_{0,\pm}$ are given by

$$\sigma_+=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad \sigma_-=\begin{bmatrix}0&0\\1&0\end{bmatrix},\ \sigma_0=\begin{bmatrix}1&0\\0&-1\end{bmatrix}. \tag{2}$$

The annihilation and creation operators, a_i and a_i^+, satisfy the usual commutation relations,

$$[a_i^+,a_j^+]=[a_i,a_j]=0,\quad [a_i,a_j^+]=\delta_{ij}. \tag{3}$$

The Hamiltonian (1) can be solved in the framework of quasi-exactly solvable problems[9] or by using numerical diagonalization method[3]. In order to obtain rotating wave approximated form of the $E\otimes\varepsilon$ Hamiltonian, we use similarity transformation by introducing the operator

$$T=\frac{\kappa}{\omega+\omega_0}\left(\sigma_+a_2^+-\sigma_-a_2\right)+\frac{\kappa}{\omega-\omega_0}\left(\sigma_-a_2^+-\sigma_+a_2\right), \tag{4}$$

and imposing the condition $|\omega\pm\omega_0|\gg\kappa$, which usually holds in the weak interaction, transformation of the Hamiltonian (1), yields that

$$\begin{aligned}\widetilde{H} \;=\; & e^THe^{-T}\approx\omega\left(a_1^+a_1+a_2^+a_2+1\right)+\omega_0\sigma_0+\kappa[(a_1+a_2)\sigma_++(a_1^++a_2^+)\sigma_-]+\\ & \left[\frac{\kappa^2}{\omega+\omega_0}\left(a_1^+a_2^++a_1a_2\right)+\frac{\kappa^2}{\omega-\omega_0}\left(a_1^+a_2+a_1a_2^+\right)\right]\sigma_0+\\ & \frac{\omega\kappa^2}{\omega^2-\omega_0^2}\left(a_2^{+2}+a_2^2+2a_2^+a_2\right)\sigma_0+\\ & \frac{\kappa^2\sigma_+\sigma_-}{\omega-\omega_0}-\frac{\kappa^2\sigma_-\sigma_+}{\omega+\omega_0}+O\left(\frac{\kappa^3}{\omega^2-\omega_0^2}\right).\end{aligned} \tag{5}$$

Since $\frac{\kappa^2}{\omega\pm\omega_0}\ll 1$ is assumed to be a small parameter, neglection of the last term confirms result;

$$\widetilde{H}\approx\omega\left(a_1^+a_1+a_2^+a_2+1\right)+\omega_0\sigma_0+\kappa[(a_1+a_2)\sigma_++(a_1^++a_2^+)\sigma_-]. \tag{6}$$

It is analytically solvable due to the neglect of the counter-rotating terms, so called RWA. Now, we turn our attention to the solution of the Hamiltonian (6). The rotation of the bosons given by the following operator

$$U=\exp\left(\frac{\pi}{4}(a_1^+a_2-a_2^+a_1\right) \tag{7}$$

provides the expressions

$$\begin{aligned}&U(a_1+a_2)U^{-1}=\sqrt{2}a_1,\quad U(a_1^++a_2^+)U^{-1}=\sqrt{2}a_1^+\\&U(a_1^+a_1+a_2^+a_2)U^{-1}=a_1^+a_1+a_2^+a_2\end{aligned} \tag{8a}$$

Under U, the Hamiltonian becomes

$$\widetilde{H}\approx\omega\left(a_1^+a_1+a_2^+a_2+1\right)+\omega_0\sigma_0+\sqrt{2}\kappa[a_1\sigma_++a_1^+\sigma_-]. \tag{9}$$

	Ground state		First excited state	
κ^2	E_{rwa}	E_{exact}	E_{rwa}	E_{exact}
0.1	0.90455	0.90442	1.85982	1.82286
0.2	0.81678	0.81595	1.73508	1.67515
0.3	0.73508	0.73277	1.62159	1.54472
0.4	0.65835	0.65371	1.51676	1.36373
0.5	0.58578	0.57798	1.41886	1.31592
0.6	0.51676	0.50498	1.32667	1.21248
0.7	0.45080	0.43429	1.23931	1.11438
0.8	0.38754	0.36557	1.15609	1.02070
0.9	0.32667	0.29856	1.07646	0.93072

TABLE I: Ground-state and first excited-state energies of the $E \otimes \varepsilon$ JT Hamiltonian.

The resultant Hamiltonian can easily be solved, because the matrix of the Hamiltonian can be decomposed in infinite dependent 2×2 blocks on the subspaces $\{|\uparrow, n_1\rangle\,|\uparrow, n_2\rangle\,, |\downarrow, n_1+1\rangle\,|\downarrow, n_2\rangle\}$, where n_1 and n_2 are the number of photons. The eigenvalue problem can be written as

$$\widetilde{H}\,|\psi\rangle = E\,|\psi\rangle \tag{10}$$

where $|\psi\rangle$ is the two component eigenstate

$$|\psi\rangle = \begin{pmatrix} c_1\,|n_1\rangle\,|n_2\rangle \\ c_2\,|n_1+1\rangle\,|n_2\rangle \end{pmatrix}, \tag{11}$$

where $c_1$and c_2 are normalization constant. Action of $\widetilde{H}$ on ψ yields the following expressions

$$\left(c_1\left(\omega\left(n_1+n_2+1\right)+\omega_0\right)+c_2\sqrt{2}\kappa\sqrt{n_1+1}\right)|n_1\rangle\,|n_2\rangle = c_1 E\,|n_1\rangle\,|n_2\rangle \tag{12a}$$

$$\left(c_2\left(\omega\left(n_1+n_2+2\right)-\omega_0\right)+c_1\sqrt{2}\kappa\sqrt{n_1+1}\right)|n_1+1\rangle\,|n_2\rangle = c_2 E\,|n_1+1\rangle\,|n_2\rangle\,. \tag{12b}$$

Eliminating c_1 and c_2 between (12a) and (12b) and solving the resultant equation for E, we obtain

$$E = (j+1)\,\omega \pm \frac{1}{2}\sqrt{8\kappa^2(n+1)+(\omega-2\omega_0)^2}. \tag{13}$$

where $j = n_1 + n_2$ total number of bosons and $n = 0, 1, 2, \cdots, 2j$. The eigenstates can be easily written by using boson operators, acting on a vacuum state$|0\rangle$;

$$|\psi\rangle = \left[c_1 a_2^{j-n} a_1^{+n}\,|0\rangle\,, c_2 a_2^{j-n} a_1^{+n+1}\,|0\rangle\right]^T. \tag{14}$$

We conclude that in weak coupling limit the oscillators does not coupled to each other and each of them oscillates with their own frequencies. We have proven that when the interaction between E ion and ε-modes are weak then $E \otimes \varepsilon$ JT Hamiltonian can be reduced to the JC model. Our formalism provides a solution of the problem which allows us to discuss the JT effects in the Dicke model.

The accuracy of the approximate eigenvalues can be checked by means of the (quasi) exact solution of the $E \otimes \varepsilon$ JT Hamiltonian. The material parameters are chosen to be $\omega = 1$ and $\omega_0 = 0$. The results are tabulated in Table 1.

The results of our study show that the eigenvalues and eigenstates of the $E\otimes\varepsilon$ JT Hamiltonian can be approximately described when the frequency ω of the oscillation larger than the interaction constant.

NON-HERMITIAN INTERACTION

It has been shown that for some purely imaginary couplings constant κ, the low-lying part of the $E \otimes \varepsilon$ JT Hamiltonian is real, although the Hamiltonian is non-Hermitian. Let us consider the Hamiltonian (9) with the imaginary coupling $\kappa = i\gamma$:

$$h = \omega\left(a_1^+ a_1 + a_2^+ a_2 + 1\right) + \omega_0\sigma_0 + i\sqrt{2}\gamma[a_1\sigma_+ + a_1^+\sigma_-]. \tag{15}$$

This Hamiltonian is not Hermitian as,

$$h^{\dagger}=\omega\left(a_1^{+}a_1+a_2^{+}a_2+1\right)+\omega_0\sigma_0-i\sqrt{2}\gamma[a_1\sigma_{+}+a_1^{+}\sigma_{-}]\neq h. \tag{16}$$

Under the parity transformation, the Pauli matrices become invariant but both the creation and annihilation operators change sign. The time reversal operator for this Hamiltonian is $T=-i\sigma_y K$ where K is complex conjugation operator. The time reversal operator changes the sign of the Pauli matrices and boson operators. It is easy to see that the Hamiltonian (15) is not PT-symmetric

$$(PT)h(PT)^{-1}=\omega\left(a_1^{+}a_1+a_2^{+}a_2+1\right)-\omega_0\sigma_0+i\sqrt{2}\gamma[a_1\sigma_{+}+a_1^{+}\sigma_{-}]\neq h. \tag{17}$$

The Hamiltonian is not PT-symmetric but it gives real spectrum. Mustafazadeh [24, 25, 26] has shown that the reality of the spectrum of non-Hermitian Hamiltonian is due to pseudo-Hermicity properties of the Hamiltonian. A Hamiltonian is called η-pseudo-Hermitian if it satisfies the following relation

$$\eta h\eta^{-1}=h^{\dagger}, \tag{18}$$

where η is a linear Hermitian operator. The Hamiltonian h and its adjoint $h^{\dagger}$ can be related to each others by the operator σ_0 and using the relation $\sigma_0\sigma_{\pm}\sigma_0^{-1}=-\sigma_{\pm}$:

$$\sigma_0 h\sigma_0^{-1}=h^{\dagger}. \tag{19}$$

Then the Hamiltonian (15) is σ_0-pseudo-Hermitian. Our Hamiltonian is also pseudo-Hermitian with respect to the parity operator. As it is shown [24] that if a Hamiltonian is pseudo-Hermitian under two different operators, η_1, η_2 then the system is symmetric under the transformation generated by $\eta_1\eta_2^{-1}$. Therefore our Hamiltonian is invariant under the symmetry generated by the combined operator, $P\sigma_0$:

$$[H,P\sigma_0]=0. \tag{20}$$

CONCLUSION

The aim of the this paper was to illustrate how the $E\otimes\varepsilon$ JT Hamiltonians can be solved by developing a transformation procedure. It has been found an approximate form of the $E\otimes\varepsilon$ JT Hamiltonian in the framework of the RWA. The resultant Hamiltonian can be solved analytically and its eigenvalues can be obtained in the closed form. We have shown that in the weak coupling limit the JT models may be recognized as the Dicke model. We have shown that when the coupling constant is imaginary the Hamiltonian is non-Hermitian but $P\sigma_0$-symmetric. We also hope to extend the method to the other JT and quantum optical systems.

* Electronic address: koc@gantep.edu.tr

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